MICRO-428: Metrology

Week Three: Elements of Statistics

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Exercise 1: Memorylessness

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- The definition of memorylessness is the following:

$$P\{Y > s + t | Y > s\} = P\{Y > t\}$$

- For the exponential distribution $Y \sim \mathrm{Expo}(\lambda)$:

$$P\{Y \leq y\} = F_Y(y) = 1 - e^{-\lambda y}$$

- Follows by definition:

$$P\{Y > s + t | Y > s\} = \frac{1 - F_Y(s+t)}{1 - F_Y(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = 1 - F_Y(t) = P\{Y > t\}$$

Exercise 2: Stationary Processes

- Let α and ω be two known constants and β a uniform RV with PDF:

$$f_U(\beta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \beta \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

- Let $X(t)$ be the RP:

$$X(t) = \alpha \cos(\omega t + \beta)$$

- Demonstrate that the RP $X(t)$ is Wide Sense stationary and, eventually, ergodic.

Exercise 2: Stationary Processes

- The WS stationary condition requires the autocorrelation function to be function only of the time variation $t_2 - t_1$:

$$K_{XX}(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\} = \frac{\alpha^2}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t_1 + \beta) \cos(\omega t_2 + \beta)\, d\beta =$$

$$= \frac{\alpha^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} [\cos(\omega t_1 + \omega t_2 + 2\beta) + \cos(\omega(t_1 - t_2))]\, d\beta = \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2))$$

- To demonstrate that the RP $X(t)$ is ergodic, we need to demonstrate that the expected value coincides with the time-average mean value, and the autocorrelation coincides with the time-average autocorrelation.

Exercise 2: Stationary Processes

- First, let us look at the expected value and time-average mean value

$$E\{X(t)\} = \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \beta)\ d\beta = \frac{\alpha}{2\pi}[\sin(\omega t + \beta)]_{-\pi}^{\pi} = 0$$

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \alpha\cos(\omega t + \beta)\,dt = \lim_{T\to\infty}\frac{\alpha}{2T\omega}[\sin(\omega t + \beta)]_{-T}^{T} =$$

$$= \lim_{T\to\infty}\frac{\alpha}{2T\omega}[\sin(\omega T + \beta) - \sin(-\omega T + \beta)] = 0$$

- Hence, we demonstrated that $\langle X(t)\rangle = E\{X(t)\}$

Exercise 2: Stationary Processes

- Then let us look at the time-average autocorrelation.

$$\mathcal{K}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \alpha\cos(\omega t + \beta)\cdot\alpha\cos(\omega(t-\tau)+\beta)\,dt =$$

$$\lim_{T\to\infty} \frac{\alpha^2}{T}\int_{-T/2}^{T/2} \frac{\cos(\omega\tau) + \cos(2\omega t - \omega\tau + 2\beta)}{2}\,dt =$$

$$\frac{\alpha^2}{2}\cos(\omega\tau) + \lim_{T\to\infty} \frac{\alpha^2}{T}\frac{1}{2}\frac{1}{2\omega}\left(\sin(\omega T - \omega\tau + 2\beta) - \sin(-\omega T - \omega\tau + 2\beta)\right) =$$

$$\frac{\alpha^2}{2}\cos(\omega\tau) = \frac{\alpha^2}{2}\cos(\omega(t_1 - t_2)) = K_{XX}(t_1, t_2)$$

- Hence, we demonstrated that $K_{XX}(t_1, t_2) = \mathcal{K}_{XX}(\tau)$, thus $X(t)$ is ergodic.

Exercise 3: Estimation using MLE

- MLE: Given a sample of n independent experiments $x_1, x_2, \dots, x_n$, and defined θ the parameters of the RV, we define the likelihood as:

$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} f_X(x_i, \theta)$$

- The MLE estimator is a value $\hat{\theta}$ such that L is maximized.

- A perfect single-photon detector (efficiency of 100%, no jitter, no DCR, etc..) is working in ultra-low photon rate regime (e.g. as a exoplanet space telescope) connected with a TDC (time-to-digital converter). The detector collects, in 50 ms, 4 photons ($t_{arrival}$ = 1, 20, 35, 38 ms). Estimate $\hat{\lambda}$.

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- The likelihood is, in this case:

$$L(t_1, t_2, t_3, t_4; \lambda) = \prod_{i=1}^{4} f_X(t_i; \lambda)$$

Exercise 3: Estimation using MLE

- Our t_i is the difference between the time of arrivals:

$$t_i = 1, 19, 15, 3 \text{ ms}$$

- Hence, given that

$$f_X(t) = \lambda e^{-\lambda t}$$

follows:

$$L(t_1, t_2, t_3, t_4; \lambda) = \lambda^4 e^{-\lambda \sum t_i}$$

- Applying the definition of MLE:

$$\ln(L) = 4\ln(\lambda) - \lambda \sum t_i$$

$$\frac{\partial}{\partial \lambda} \ln(L) = \frac{4}{\lambda} - \sum t_i \quad \rightarrow \quad \hat{\lambda} = \frac{4}{\sum t_i}$$

Homework 1: Estimation using MLE

- The measured fluorescence lifetime curve is modeled as the convolution between the exponential function and the instrument response function (IRF). Considering that we measure fluorescence lifetime with an ideal setup (IRF is the Dirac function), calculate the maximum likelihood estimator of fluorescence lifetime given the arrival time of N photons.

Homework 2: Random Walk

- The random walk is a random process which can be used to model the path resulting from random steps to the left ($X_j = 1$) or to the right ($X_j = -1$) starting from the position 0.

- Show that the variance of the random walk is maximum for

$$P\{X_j = 1\} = p = 0.5$$

HINT: the final position of N-step random walk is the sum of X_j :

$$Y = \sum_{j=1}^{N} X_j$$